

## NONLINEAR STABILITY OF SOLID PROPELLANT COMBUSTION

F.J. Higuera and A. Linan

1. INTRODUCTION. The stability and transient response of solid propellant combustion has received considerable attention in the literature, using, after Zeldovich (1942 and 1964) and Denison and Baum (1961), the quasisteady approximation for the gas phase. This approximation results from the small value of the ratio of gas-to-solid densities.

The analysis of the stability of a simple model of solid propellant combustion is given here. The model includes a surface pyrolysis reaction and a gas phase reaction, assuming the gas phase combustion process to be quasisteady and quasiplanar. The quasiplanar assumption results from the moderately small value of the ratio of thickness of the gas-to-solid transport zones, or from large values of the effective nondimensional activation energy of the gas phase reaction.

Two parameters,  $\epsilon_g^{-1}$  and  $\epsilon_s^{-1}$ , characterizing the sensitivity of the gas phase and pyrolysis reaction with temperature, enter into the linear stability analysis and a third parameter appears in the nonlinear analysis. Travelling waves of non-zero wavenumber are found on the stability boundary  $R(\epsilon_g, \epsilon_s) = 0$ . These results are quoted in Section 2.

A nonlinear stability analysis, close to the stability boundary, is carried out in Section 3 for the case of practical

importance corresponding to large values of the gas phase activation energy  $\epsilon_g^{-1}$ . In this limit the characteristic transverse length and response time become large compared to the thickness of the transport zone in the solid and the residence time in this zone, respectively. Three different time scales are found from the linear stability analysis. A system of nondispersive waves is found in the shorter time scale; these waves are weakly dispersive and then dispersive effects can be counterbalanced by small nonlinear effects in a second time scale. The resulting problem coincides with that associated with the propagation of gravity wave in shallow water, and the Korteweg-deVries equation describes the evolution of waves travelling in any direction. Under very general conditions the asymptotic state of a wave train can be predicted by the inverse scattering method and consists in a series of solitons and a residual wave train. The analysis corresponds to an infinite system; results for an interface of finite size have been obtained by Margolis (1985) for another two different models of combustion of solids.

Finally, the growth or decay effects predicted by the linear analysis appear in the longest time scale, resulting in a small perturbation to the Korteweg-deVries equation that includes new nonlinear terms in addition to those coming from the linear theory. The perturbation changes slowly the amplitude and velocity of each of the solitons toward a defined value or zero, depending on the initial size of the soliton and the values of the parameters. The analysis for a single soliton is carried out in Section 4.

In the last section of equations describing the slow evolution of a nonlinear modulated wavetrain is obtained from a pseudovariational formulation. The hyperbolic character of these equations is not altered by the perturbations due to the real part of the growth rate.

2. FORMULATION, STEADY STATE AND RESULTS FROM THE LINEAR STABILITY ANALYSIS. We analyze the non-linear stability of the combustion of a solid propellant subject to nonplanar perturbations, for the simple model of Denison and Baum (1961), which includes a surface heterogeneous reaction and a gas phase reaction. Both are supposed to be global one-step reactions following Arrhenius laws, the second one with large activation energy. No further reactions occur inside the homogeneous solid and the transport coefficients and specific heats are constant in each phase.

A relation of the form

$$m = m_s(T_s) = B \exp(-E_s/RT_s) \quad (1)$$

for the mass flux  $m$  crossing the interface, as a function of the interface temperature  $T_s$ , is assumed to represent the surface reaction.  $E_s$  is the activation energy for this reaction, and  $B$  is a preexponential constant.

The reaction zone in the gas phase is a thin layer behind a transport zone; from the analysis of this layer and its matching to the transport zone a second relation

$$m_f(T_f) = C \exp(-E/2RT_f) \quad (2)$$

between the mass flux and the flame temperature  $T_f$  is obtained.  $E$  is the activation energy for the gas phase reaction, and  $E/RT_f$  is assumed to be large. If the gas phase transport zone is assumed to be planar and steady,  $m_f(T_f) = m$ .

The thickness of the transport zone in the gas and the conduction zone in the solid are given by  $\delta_g \sim \lambda_g/mc_p$  and  $\delta_s \sim \lambda_s/mc_s$ , respectively, as results from the balance between convective transport due to the recession of the interface and conduction. It turns out that  $\delta_g \ll \delta_s$ , due to the

fact that the thermal conductivity of the gas,  $\lambda_g$ , is much smaller than that of the solid,  $\lambda_s$ . The residence times in these two zones are  $\tau_g \sim \rho_g \lambda_g / m^2 c_p$  and  $\tau_s \sim \rho_s \lambda_s / m^2 c_s$ , and  $\tau_g \ll \tau_s$ , because of the small value of the gas-to-solid density ratio. The analysis that follows is addressed to the response of the solid phase, and the times involved are of the order of  $\tau_s$ ; in addition the characteristic transverse length of the perturbations that can grow is of the order of  $\delta_g$ . Both facts taken together allow a quasisteady and locally one-dimensional treatment of the gas phase; so that in particular, the relations (1-2) are still valid in the transient analysis. The problem is reduced to solve the heat conduction equation in the interior of the solid, and to determine the shape of the perturbed interface. Most of the analysis can be directly applied to other combustion models. There is a steady state solution in which the variables depend only on the distance  $\zeta$  to the planar interface,  $\zeta=0$ ; with the solid lying at  $\zeta < 0$ . The steady flame temperature  $T_{fo}$  is given by

$$c_p T_{fo} = c_s T_\infty + Q \quad (3)$$

Where  $Q$  is the heat released per unit mass by both reactions and  $T_\infty$  is the temperature of the solid far from the interface. Here and in the following the subscript  $o$  is used for the magnitudes in the steady state. Eq. (3) is an overall energy balance. The steady mass flux  $m_o$  results then from (2) and the steady interface temperature  $T_{so}$  from (1). The final step would be to solve the energy conservation equation in the solid to find the steady temperature profile.

In a reference frame moving relative to the solid with the steady recession velocity the general unsteady problem can be formulated as follows

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial \zeta} = v^2 \theta \quad (4)$$

$$\zeta \rightarrow -\infty : \theta = 0 \quad (5)$$

$$\zeta = X_s : \begin{cases} \theta = [1 + \frac{\omega}{\gamma-1} \ln \mu] / (1 - \omega \ln \mu) \\ \frac{\partial \theta}{\partial n} - \mu \theta = -\epsilon_g \mu \ln \mu / (1 - \omega_g \ln \mu) \end{cases} \quad (6)$$

where  $\theta = (T - T_\infty) / (T_{so} - T_\infty)$  is the nondimensional temperature and  $\mu = m/m_0$  is the nondimensional mass flux. The distances have been referred to  $\lambda_s/m_0 c_s$  and the time to  $\rho_s \lambda_s / m_0^2 c_s$ . The parameters that appear are

$$\omega = RT_{so}/E_s, \quad \gamma = T_{so}/T_\infty, \quad \omega_g = 2RT_{fo}/E, \\ \epsilon_g = \frac{c_p}{c_s} \frac{2RT_{fo}^2}{E(T_{so} - T_\infty)} \quad \text{and} \quad \epsilon_s = \frac{RT_{so}^2}{E_s(T_{so} - T_\infty)} \quad (8)$$

The shape of the interface is given by  $\zeta = X_s(y, z, t)$ , where  $y$  and  $z$  are transverse coordinates, and the nondimensional mass flux can be written as

$$\mu = (1 - X_{st}) \sqrt{1 + X_{sy}^2 + X_{sz}^2} \quad (9)$$

Finally  $n$  is normal to the interface. The first condition at the interface comes from (1), and the second one results from an energy balance between the solid side of the interface and the gas behind the flame, after (2) is used to eliminate the flame temperature.

In order to analyze the linear stability of the steady state, small perturbations are considered with a dependence on time and the transversal coordinates through a factor of the form  $\exp(\Omega t + iky)$ . (Nothing new comes from the dependence on

on the other transverse coordinate in the linear analysis.) The results, after the whole problem is linearized, lead to the compatibility condition, or dispersion relation, of the type

$$\begin{aligned} \epsilon_s^2 \Omega^3 + \{\epsilon_s^2 k^2 + (1+\epsilon_g) \epsilon_s - (1-\epsilon_g)^2 \Omega^2 \\ + (2 \epsilon_s k^2 + \epsilon_g) \Omega + k^2\} = 0 \end{aligned} \quad (10)$$

which has been written as a cubic equation for  $\Omega$ . It is worth noticing that only two parameters,  $\epsilon_s$  and  $\epsilon_g$ , appear in (10); their inverses characterize the temperature sensitivity of the interface and gas phase reactions (see Higuera and Linan, 1982).

Due to the large value of the activation energy for the gas phase reaction, the parameter  $\epsilon_g$  can take very small values, and this is the case considered in what follows. The stability limit for  $\epsilon_g=0$  is given by  $\epsilon_s=2$  and the steady state is unstable when  $\epsilon_s < 2$ . For small values of  $\epsilon_g$  the perturbations that can grow correspond to large wavelengths,  $k=\delta K$  with  $\delta=\epsilon_g^{1/2} \ll 1$  and  $K \sim 1$ . The dispersion relations close to the stability limit can be written in the form (with  $\epsilon_s = (1+\delta^2 \alpha)$  and  $\alpha \sim 1$ )

$$\Omega = \pm i \delta K + i \delta^3 K (\alpha + 2k^2) - \delta^4 \{1/2 + 2k^2(\alpha + ik^2)\} + \dots$$

and is sketched in Fig. 1.

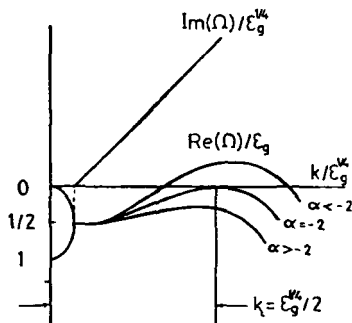


Fig. 1

The role of the imaginary part, which is larger than the real part, can be interpreted as follows: Every heated band of the solid is almost independent of its neighbours, because the wavelength of the perturbations is large. In first approximation the changes in the temperature distribution inside the solid when the burning rate increases are due to two effects (Fig. 2). On one hand the whole temperature profile is squeezed against the interface and on the other the temperature rises because, at the interface, it is an increasing function of the burning rate. When  $\varepsilon_s$  is close to 2, the second effect dominates over the first one and therefore the temperature in most of the band increases with the burning rate,  $-X_{st}$ , and so does the heat content. Now the origin of these changes in the heat content and the burning rate is the heat conduction tangential to the interface, coupling different heated bands of the solid, which is due to temperature differences at a fixed depth when the interface is deformed, see Fig. 2. The heat flux is proportional to the slope  $X_{sy}$  and the net flux received by a given band goes like  $-X_{sy}$ ; this being the origin of the temperature changes, one obtains the relation

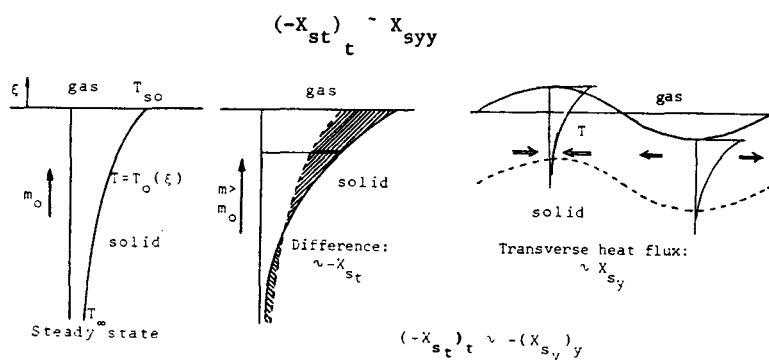


Fig. 2

accounting for the leading order of the expansion (11). The next term in (11) involves a correction proportional to  $K^3$  representing dispersion. This comes from the fact that the

temperature distributions in the last sketch in Fig. 2, responsible for the heat conduction, are not exactly equal to those of the steady state profile, but include a small change proportional to the second derivative of the front position and, in addition, the time derivatives in the heat conduction equation provide other small term proportional to  $X_{s\text{ttt}}$  in the heat content of the band. Both effects lead to a small correction, proportional to the fourth derivative of the front position, in the previous estimate, accounting, after the square root is taken, for the  $K^3$  term in (11).

**3. DERIVATION OF THE NONLINEAR EVOLUTION EQUATIONS.** The three different time scales appearing in the dispersion relation are explicitly introduced in the nonlinear analysis through a multiscale method in which the functions depend on the variables

$$\zeta, \eta = \delta y, \tau = \delta t, T = \delta^3 t \text{ and } T' = \delta^4 t \quad (12)$$

where only one variable is introduced in the plane of the interface because, for the moment, the analysis is restricted to onedimensional waves.

The only new piece of information to be introduced is the order of magnitude of the (finite) perturbations. The clue for this choice comes from the fact that nonlinearity must counterbalance dispersion, because otherwise the amplitude of any initially localized perturbation would decay as  $1/\sqrt{E}$ , before the growth predicted by the linear analysis can have any effect. In order to get this balance one must use expansions of the type

$$\begin{aligned} X_s &= \delta X + \delta^2 X^{(2)} + \dots \\ \theta - \theta_0 &= \delta^2 \theta^{(2)} + \delta^3 \theta^{(3)} + \dots \end{aligned} \quad (13)$$

into (4-7).



The results of the multiscale analysis are simply quoted, skipping over the intermediate steps.

In the faster time scale the waves on the interface satisfy the wave equation

$$X_{\tau\tau} = X_{\eta\eta} \quad (14)$$

and from here on only waves travelling to the left will be considered, so that

$$X = X(\eta + \tau, T, T') \quad (15)$$

Waves travelling to the right can be accounted for in a similar way and, for perturbations that are initially of bounded support, the final results to be obtained are valid in spite of the nonlinear interaction between both families of waves.

By proceeding with the multiscale analysis the following equation results for the evolution in the intermediate scale  $T$ .

$$X_{\eta T} = -\alpha X_{\eta\eta} + 2 X_{\eta\eta\eta\eta} + 2(\omega - 2) X_{\eta} X_{\eta\eta} \quad (16)$$

where  $\bar{\eta} = \eta + \tau$  and  $\omega$  is a third parameter, defined before, that does not appear in the linear stability analysis, and takes values between 0 and 2, when  $\epsilon_s$  is close to 2. Eq. (16) can be written in the rescaled variables

$$x = -2^{-1/3}(\bar{\eta} - \alpha T), \quad v = 2^{1/3}(2 - \omega)X/6, \quad W = v_x \quad (17)$$

in the form

$$W_T + 6WW_x + W_{xxx} = 0 \quad (18)$$

which is a standard form of the Korteweg-deVries equation, so that up to here the resulting problem coincides with that

associated with the propagation of gravity waves in shallow water. It can be seen that the nonlinear term provides the typical hyperbolic steepening of the profiles, counterbalancing the dispersive behaviour coming from the third derivative.

Finally the growth or decay effects predicted by the linear analysis appear in the slow time scale. The results can be summarized in the form of a perturbation to the K-dV equation, which takes the form

$$W_T + WW_x + W_{xxx} = \delta \Gamma(W) \quad (19)$$

with

$$\begin{aligned} \Gamma(W) = & -2^{5/3} W_{xxxx} + 2^{1/3} \alpha W_{xx} \\ & - W/2 - 3 \cdot 2^{2/3} \frac{1-\omega}{2-\omega} (W^2)_{xx} \end{aligned} \quad (20)$$

4. PERTURBATION OF A SOLITON. Let us begin by considering the unperturbed equation (18). This is a prototype for the long wavelength behaviour of many systems; it was first found in the analysis of shallow water waves (Korteweg and deVries, 1895). It is well known that (18) can be exactly solved by the inverse scattering transform, under fairly general initial conditions.

The idea of the inverse scattering transform is to associate the linear eigenvalue problem

$$\phi_{xx} + [\lambda^2 + W(x, T)] \phi = 0, \quad -\infty < x < \infty \quad (21)$$

with the nonlinear equation (18). The solution  $W(x, T)$  that we are looking for appears here as a potential in the Schrodinger equation (21) and  $T$  plays the role of parameter. It is assumed that  $W$  tends to zero sufficiently fast when  $x \rightarrow \pm \infty$  and that  $\int_{-\infty}^{\infty} (1+|x|)|W|dx$  is finite for all  $T$ . In these

conditions all the real values of  $\lambda$  constitute spectrum of (21), and it is possible to find eigenfunctions  $\phi(x, \lambda, T)$  normalized according to the conditions

$$\phi(x, \lambda, T) \rightarrow \begin{cases} e^{-i\lambda x}, & x \rightarrow -\infty \\ a(\lambda, T) e^{-i\lambda x} + b(\lambda, T) e^{i\lambda x}, & x \rightarrow \infty \end{cases} \quad (22)$$

which represent a wave travelling from the right; on arriving to the region where  $W$  is different from zero this wave is partially transmitted and partially reflected backward;  $b/a$  is known as the transmission coefficient. The function  $a(\lambda)$  admits an analytic extension into the upper half complex  $\lambda$ -plane and its zeros on the imaginary axis ( $\lambda_k = iV_k$ ,  $V_k > 0$ ) are the discrete eigenvalues of (21). The corresponding eigenfunctions  $\phi_k$  tend to zero as  $\exp(V_k x)$  when  $x \rightarrow -\infty$  and as  $b_k \exp(V_k x)$  when  $x \rightarrow \infty$ , where  $b_k$  is the analytic extension of  $b(\lambda)$  to  $\lambda_k$ . As is well known, (Gelfand and Levitan, 1951), the set of scattering data

$$s = b/a(\lambda), \lambda \text{ real}; \lambda_k = iV_k, \gamma_k = b_k/a'_k \quad (23)$$

where  $a'_k = (\partial a / \partial \lambda)_{\lambda_k}$ , contains sufficient information to reconstruct the potential  $W$ . The second point is that when  $W(x, T)$  is a solution of the K-dV equation it is possible to find ordinary differential equations for the evolution of the various spectral components, separated from each other (see, p.e., Newell, 1978). The resolution of (18) proceeds therefore along the following lines: given the initial profile  $W(x, 0)$ , one can state (21) and find the set of initial scattering data. These provide the initial conditions for the evolution equations for the scattering data and, once these equations have been solved, the solution  $W(x, T)$  can be constructed from them.

There is a soliton associated with each eigenvalue of the discrete spectrum. One of these solitons taken alone is a

solution of (18) of the form

$$W = 2V^2 \operatorname{sech}^2[V(x-4V^2T)] \quad (24)$$

where  $iV$  is the only discrete eigenvalue of (21) for this  $W(x,T)$ . In the original variables the solution (24) takes the form

$$\bar{x} = 2^{2/3} \cdot 6 \frac{V}{2-W} \operatorname{Th} \left[ V \frac{(n+z-\alpha T)}{2^{1/3}} \right] \quad (25)$$

which represent a step on the interface travelling toward the left with velocity  $1-\delta^2 (\alpha-2^{1/3} \cdot 4V^2)$ .

Asymptotically, when  $T \rightarrow \infty$ , the solution of (18) tends to a series of solitons plus a residual wave train. The number and sizes of the solitons depend on the initial conditions and, as can be seen, the bigger the soliton is the faster it travels, so that, after some time, they become ordered and separate away from each other.

When  $W(x,T)$  evolves according to the perturbed K-dV equation (19), it is no longer possible to separate the evolution equations for the various components of the scattering set. Nevertheless, if  $\delta \ll 1$ , the change of scattering data can be obtained through some perturbation method applied to the coupled system, which can be written as, (newell, 1978).

$$\frac{d}{dT} \left( \frac{b}{a} \right) = 8i\lambda^3 \frac{b}{a} - \frac{\delta}{2i\lambda a} \int_{-\infty}^{\infty} \Gamma(W) \phi^2 dx \quad (26)$$

$$\frac{d}{dT} (\lambda_k) = \frac{\delta}{2i\lambda_k \gamma_k a_k} \int_{-\infty}^{\infty} \Gamma(W) \phi_k^2 dx \quad (27) \quad \left( \right.$$

$$\frac{d}{dT}(\gamma_k) + \left(\frac{a_k''}{a_k'} + \frac{1}{\lambda_k}\right) \gamma_k \frac{d\lambda_k}{dT} = 8i\lambda_k^3 \gamma_k \frac{\delta}{2i\lambda_k a_k'^2} \int_{-\infty}^{\infty} \Gamma(W) \left(\frac{d\phi^2}{d\lambda}\right)_{\lambda_k} dx \quad (28)$$

It is not clear how the general solution of these equations would look like. Here we are going to consider only an initial perturbation consisting in a single soliton and are going to look for its slow evolution taking the solution in the form

$$W = 2V^2 \operatorname{sech}^2 [V(x-\sigma)] \quad (29)$$

where  $V$  is a slowly changing variable and  $\sigma = \int 4V^2 dT$  is a phase. This should be an important case, according to what was said before, and even in this amplified approach there appear complicated features, which are not going to be addressed here. This concern, in particular the possible birth of secondary solitons from the continuous spectrum, which always produces an oscillatory tail behind the main step. The eigenfunction associated with  $\lambda_1 = iV$  and the scattering data for (29) can be written as

$$\phi_1 = \phi(\lambda = iV) = \frac{b_1}{2} e^{-V\sigma} \operatorname{sech} [V(x-\sigma)] \quad (30)$$

$$a(\lambda) = \frac{\lambda - iV}{\lambda + iV}, \quad b_1 = b_1(0) e^{V\sigma}, \quad \gamma_1 = 2iV e^{2V\sigma}$$

Using now these and similar expressions to evaluate the right hand sides of (26-28) a closed system would result. In particular

$$\frac{dV}{dT} = -\frac{\delta V}{4} \left\{ \frac{4}{3} + \frac{32}{15} 2^{1/3} \alpha V^2 + \frac{16^2}{21} \left(1 - \frac{6}{5} \frac{1-\omega}{2-\omega}\right) 2^{2/3} V^4 \right\} \quad (31)$$

is obtained from the central equation (27), which becomes uncoupled from the others. In so far as this perturbation approach is valid Eq. (31) describes the slow evolution of the soliton. There is a critical value  $\alpha_c$ , depending on  $\omega$ , such that when  $\alpha > \alpha_c$  the initial amplitude of the soliton decreases,

continuously until it disappears. Whereas if  $\alpha < \alpha_c$  the asymptotic state will depend on the initial size of the soliton; which will disappear if it is smaller than the value  $V_L$  in the sketch in Fig. 3, and will tend to  $V_\infty$  otherwise. The critical value of  $\alpha$  is

$$\alpha_c(\omega) = -\frac{10}{\sqrt{7}} \left[ 1 - \frac{6}{5} \frac{1-\omega}{2-\omega} \right]^{1/2} \quad (32)$$

and is plotted in Fig. 4 together with  $V_c(\omega)$

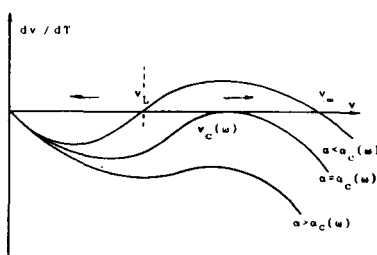


Fig. 3

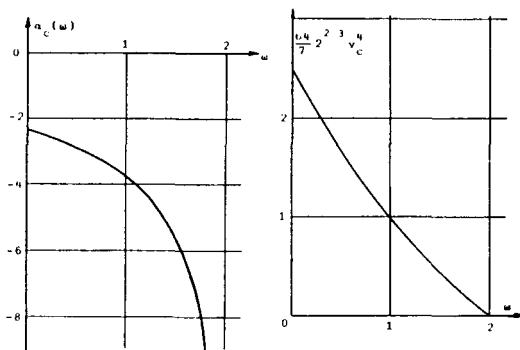


Fig. 4

Up to this point the waves are restricted to travel in a fixed direction. We are now going to comment briefly on the effects coming from the geometry in the propagation of a solitary wave on the interface, when these effects are of the same order as those coming from the real part of  $\Omega$  in the dispersion relation (leading to the right hand side in (19)). This accounts for sufficiently small curvatures of the front;

when the curvature is larger the geometrical effects dominate and determine the evolution of the wave without any significant influence from the dispersion or the growth predicted by the linear stability analysis. Let the successive positions of the front be given by  $f_1(\eta, \zeta) - \tau = 0$  and let  $f_2(\eta, \zeta) = \text{const.}$  be the orthogonal trajectories or rays. Both families taken together define an orthogonal curvilinear coordinate set (as sketched in Fig. 5). The length of an elementary arc can be written as  $ds^2 = S^2 df_1^2 + B^2 df_2^2$ , where the metric coefficients satisfy the geometrical condition

$$\frac{\partial}{\partial f_1} \left( \frac{1}{S} \frac{\partial B}{\partial f_1} \right) + \frac{\partial}{\partial f_2} \left( \frac{1}{B} \frac{\partial S}{\partial f_2} \right) = 0 \quad (33)$$

In order to determine the functions  $f_1$  and  $f_2$  the best procedure is to write down equations for the angle  $\theta$  between the rays and a fixed direction in the plane of the interface. As can be seen from Fig. 5,  $\theta$  is the solution of

$$\frac{\partial \theta}{\partial f_2} = \frac{1}{S} \frac{\partial B}{\partial f_1} \quad (34)$$

$$\frac{\partial \theta}{\partial f_1} = - \frac{1}{B} \frac{\partial S}{\partial f_2}$$

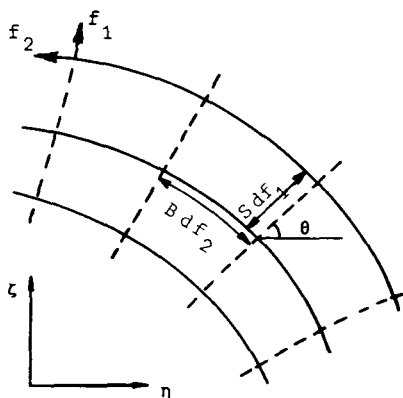


Fig. 5

To close the problem the following two relations must be taken into account.

$$S = 1 - \delta^2 \alpha + 4 \cdot 2^{1/3} \delta^3 v^2 \quad (35)$$

$$\frac{\partial v / \partial f_1}{S^2} = -\frac{\delta^3 v}{4} \left\{ \frac{4}{3} + \frac{32}{15} 2^{1/3} \alpha v^2 + \frac{16^2}{21} \left( 1 - \frac{6}{5} \frac{1-\omega}{2-\omega} \right)^{2/3} v^4 \right\} - \frac{2}{3} \Gamma_1 v \quad (36)$$

where

$$\Gamma_1 = \frac{\partial B / \partial f_1}{2SB} \quad (37)$$

The first of these is simply the expression of the normal wave velocity in terms of its amplitude, and the second is a generalization of (32), accounting also for the slow variation of the breadth  $B$  along the "channel" between two rays, which leads to the final term. A term of this form appears in the theory of shallow water waves in a slowly varying channel (see p.e. Miles, 1979). A definite approximation is involved here because of course the rays are not confining walls.

The system (33-37) is a closed problem whose solution would give  $\theta$  and the local wave amplitude as functions of  $f_1$  and  $f_2$ . Finally,  $f_1$  and  $f_2$  can be tied to the fixed directions  $\eta$  and  $\zeta$  on the interface through the solution of

$$\begin{aligned} \partial f_1 / \partial \eta &= \cos \theta / S & \partial f_1 / \partial \zeta &= \sin \theta / S \\ \partial f_2 / \partial \eta &= -\sin \theta / B & \partial f_2 / \partial \zeta &= \cos \theta / B \end{aligned} \quad (38)$$

as can be seen from Fig. 5.

As a specific example let us consider the case of a symmetrical wave travelling away from a point. In this case the final term in (36) takes the form  $-v/3r$ , where  $r$  is the distance to the source of the wave. The analysis applies when  $r \sim \delta^3$  and, as can be seen, the geometrical divergence tends



to decrease the amplitude of the wave at a rate that decreases as the wave evolves and  $r$  increases. So the divergence changes the limiting value  $V_L$  found before for planar waves and some waves initially larger than this limiting value can end up decaying and disappearing.

5. MODULATION APPROACH. There is a second type of travelling wave solutions of the K-dV equation, cn wave, which are periodic travelling waves of the form  $W=W(\theta)$  with  $\theta = kx - \Omega T$ . Here  $\Omega$  and  $k$  are constants and  $W(\theta)$  is a  $2\pi$  periodic function. (In fact the solitary wave can be obtained as the infinite wavelength limit from the cn wave). It is possible to obtain evolution equations for the slowly varying amplitude and frequency of a cn wave under the effect of the perturbation in the right hand side of (19), by taking advantage from one (appropriate) conservation law of the infinite number that there exist for the unperturbed K-dV equation. A somewhat more general approach comes from modulation theory, which accounts for the evolution of slowly changing waves including their slow dependence on spatial coordinates in addition to their temporal changes. The solution is taken in the form

$$V = \Psi + \phi(\theta) \quad (39)$$

whose  $\Psi = \Psi(x', T')/\delta$  is the slowly varying mean level of the interface and  $\theta = \theta(x', T')/\delta$  is a phase variable.

$\phi(\theta) = \phi(\theta + 2\pi)$  represents the oscillatory part of the wave, and  $x' = \delta x$  and  $T' = \delta T$  are the slow variables. Now the local values

$$\beta = \Psi_x, \quad \gamma = -\Psi_T, \quad k = \theta_x \quad \text{and} \quad \Omega = -\theta_T \quad (40)$$

still depend slowly on space and time. In particular,  $W = \beta + k\phi_0$ . Carrying (40) into (18), the following equations results for the leading order in an asymptotic expansion in powers of  $\delta$

$$k^2 W_{\theta\theta\theta} + 6WW_{\theta} - UW_{\theta} = 0 \quad (41)$$

where  $U = \Omega/k$ . After two integrations the expression

$$k^2 W_{\theta}^2 + 2W^3 - UW^2 + 2BW - 2A = 0 \quad (42)$$

is obtained. Here  $A$  and  $B$  are slowly varying functions. A further integration allows to calculate  $W(\theta)$  in terms of  $A$ ,  $B$ ,  $U$  and  $k$ . The equations describing the slow evolution of these variables, as well as  $\Psi$ , would result from the appropriate resolvability conditions on the higher order terms in the multiscale analysis. Instead of going directly into these higher order approximations on the equation itself, it proves to be more efficient, for the unperturbed  $K$ -dV equation, to take into account the existence of a variational formulation, and to carry the expressions (39-42) into the lagrangian density, taking variations only after the fast variables have been suppressed by averaging the lagrangian over a cycle of the oscillation. The perturbation in the right hand side of (19) destroys the variational formulation, however, it is still useful to apply the averaging method on a pseudo-variational principle, in which the lagrangian depends on the functions to be varied and the solution of the problem. This can be used to describe irreversible processes and was introduced by Jimenez and Whitham (1976) as an extension of the modulation approach for lagrangian systems (see Whitham, 1974). The perturbation in the right hand side of (19) can be written as

$$\delta \Gamma(V) = \delta \frac{\partial}{\partial x} \{ \Gamma'(V) \} \quad (43)$$

where

$$\Gamma'(V) = -2^{5/3} V_{xxxx} + 2^{1/3} \alpha V_{xx}^{-V/2-3} \cdot 2^{2/3} \frac{1-\omega}{2-\omega} (V_{xx}^2) \quad (44)$$

and one possible pseudo-variational principle leading to (19) results from the lagrangian density

$$L = -\frac{1}{2} V_T V_x - V_x^3 + \frac{1}{2} V_{xx}^2 + \delta V_x \tilde{\Gamma}' \quad (45)$$

where the tilde means that  $\Gamma'$  is kept constant when variations are taken. Now if (39) is taken into (45) and (41-42) are used, the averaged lagrangian

$$\begin{aligned} \mathcal{L} = \langle L \rangle = kZ(A, B, U) = & \beta B + \frac{1}{2} \beta \gamma - \frac{1}{2} U \beta^2 - A \quad (46) \\ & - \frac{1}{2} \beta \tilde{\Psi} - \delta k (\theta - \tilde{\theta}) \langle \phi_{\theta\theta} \tilde{\Gamma}' \rangle \end{aligned}$$

results, where

$$Z = \frac{1}{2\pi} \oint [2A - 2BW + UW^2 - 2W^3]^{1/2} dW \quad (47)$$

and

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} (\cdot) d\theta \quad (48)$$

The averaged variational equations coming from (46)

are

$$\begin{aligned} \mathcal{L}_B = 0, \quad -\frac{\partial \mathcal{L}}{\partial T'} + \frac{\partial \mathcal{L}}{\partial x'} - \mathcal{L}_\Psi = 0, \quad \mathcal{L}_A = 0 \quad \text{and} \\ \frac{\partial \mathcal{L}}{\partial T'} - \frac{\partial \mathcal{L}}{\partial x'} - \mathcal{L}_\theta = 0 \end{aligned} \quad (49a)$$

together with the consistency relations

$$\frac{\partial \beta}{\partial T'} + \frac{\partial \gamma}{\partial x'} = 0 \quad \text{and} \quad \frac{\partial k}{\partial T'} + \frac{\partial \Omega}{\partial x'} = 0 \quad (49b)$$

which result from the definitions (40).

After a certain amount of algebra these expressions can be transformed into the system

$$\frac{\partial Z_B}{\partial T'} + U \frac{\partial Z_B}{\partial x'} + Z_A \frac{\partial B}{\partial x'} = -Z_B/2 \quad (50a)$$

$$\frac{\partial Z_U}{\partial T'} + U \frac{\partial Z_U}{\partial x'} - Z_A \frac{\partial A}{\partial x'} = k \langle \phi_{00} \rangle \Gamma' - Z_B^2/2 Z_A \quad (50b)$$

$$\frac{\partial Z_A}{\partial T'} + U \frac{\partial Z_A}{\partial x'} - Z_A \frac{\partial U}{\partial x'} = 0 \quad (50c)$$

describing the slow evolution of the wave train, whereas the other variables are given by

$$k = 1/W_A, \quad \Omega = U/W_A, \quad \beta = -W_B/W_A \quad \text{and}$$

$$\gamma = -UW_B/W_A - B \quad (51)$$

in terms of A, B and U.

A few general comments may be made about (50). First it can be seen that it is a hyperbolic system, and can be written in characteristic form if the roots p, q and r of the cubic in the integrand of (47) are introduced as new variables in place of A, B and U. The hyperbolic character of (50) means that a modulated wavetrain is stable in the Whitham sense (Whitham 1974). The second fact to notice is that the perturbation in the right hand side of (19) is reflected only in the right hand sides of (50), and these terms do not involve derivatives of the variables, so that they do not modify the local slope of the characteristic curves in the (x', T') plane. Their main effect is to destroy the Riemann invariants that, otherwise, would be the combinations p+q, q+r and r+p. Finally, it must be remarked the coupling between the slow variations in wave amplitude and phase and the slow variations in the mean level of the interface.

## ACKNOWLEDGEMENT

We acknowledge the partial support for this research by the Spanish CAICYT under Project 2291-83.

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- E.T.S. Ingenieros Aeronauticos. Universidad Politecnica de Madrid. Pza Cardenal Cisneros 3, 28040 Madrid. Spain.